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UNSTEADY TEMPERATURE FIELD IN A PLANE BOUNDED WITHIN BY A NONCIRCULAR CONTOUR

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We consider a problem on unsteady propagation of heat in a plane infinite region bounded within by a convex contour Γ, on which a constant temperature u_0 is maintained. The initial temperature within the region is assumed equal to zero. Let the equation of the contour Γ in polar coordinates have the form $r = a\gamma(\varphi)$, where a is the characteristic linear dimension of the problem.

The problem reduces to the solution of the following differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{\varkappa}\frac{\partial u}{\partial \tau}, \qquad r > a\gamma(\varphi) \tag{1}$$

with the initial and boundary conditions

$$\begin{aligned} &u \to 0 \ \text{ for } \tau \to +0,\ r > a\gamma(\varphi) \\ &u \to u_0 \ \text{ for } r \to a\gamma(\varphi),\ \tau > 0 \\ &u \to 0 \ \text{ for } r \to \infty,\ \tau > 0 \end{aligned} \tag{2}$$

We seek the solution of the problem (1), (2) for small intervals of time. Let u^* denote the Laplace transform of u with respect to τ, i.e.

$$u^* = u^*(s) = \int_0^\infty u e^{-s\tau}\, d\tau$$

Introducing the dimensionless radius $\rho = r / a$ and using the properties of the Laplace transforms, we can write (1) and (2) in the form

$$\varepsilon^2\left(\frac{\partial^2 u^*}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u^*}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u^*}{\partial \varphi^2}\right) - u^* = 0, \qquad \rho > \gamma(\varphi) \qquad \left(\varepsilon^2 = \frac{\varkappa}{sa^2}\right)$$

$$\begin{aligned} &u^* \to u_0 / s \ \text{ for } \rho \to \gamma(\varphi) \\ &u^* \to 0 \ \text{ for } \rho \to \infty \end{aligned} \tag{3}$$

Since we solve the problem (1), (2) for small time intervals, we seek the asymptotics of the solution of (3) for $s \to \infty$, assuming the parameter ε^2 is small; the latter appears in (3) as a multiplier of the higher order derivatives. In the present case we have a regular degeneration of the boundary value problem which was studied in detail in [1, 2]. The solution of (3) is of the boundary layer type and decays rapidly on moving away from the boundary Γ.

Let us introduce a new variable $t = [\rho - \gamma(\varphi)] / \varepsilon$ corresponding to stretching the neighborhood of Γ by $1/\varepsilon$ times. Using the variables t and φ we can rewrite the operator appearing in the left-hand side of (3) in the form

$$L_\varepsilon = \varepsilon^2\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right) - 1 = \sum_{k=0}^\infty \varepsilon^k M_k \tag{4}$$

$$M_0=\frac{\partial^2}{\partial t^2}-1, \qquad M_{k+1}=(-1)^k\frac{t^{k-1}}{\gamma^{k+1}}\left[t\frac{\partial}{\partial t}-k\frac{\partial^2}{\partial\varphi^2}\right] \tag{5}$$

$$(k=0,\ 1,\ 2,\ \ldots)$$

We seek the solution of the problem (3) in the form

$$u^*=\frac{u_0}{s}\sum_{l=0}^{\infty}\varepsilon^l u_l{}^* \tag{6}$$

By virtue of the linearilty of the operator L_ε

$$L_\varepsilon u^*=\frac{u_0}{s}\sum_{k,\,l=0}^{\infty}\varepsilon^{k+l}M_k u_l{}^*=\frac{u_0}{s}\sum_{m=0}^{\infty}\varepsilon^m\left(\sum_{k=1}^{m}M_{m-k+1}u^*_{k-1}\right) \tag{7}$$

Equating to zero the terms in (7) accompanying various powers of ε and using (3), we obtain the following recurrent sequence of the boundary value problems:

$$M_0u_0{}^*=0, \quad u_0{}^*|_{t=0}=1, \quad u_0{}^*|_{t\to\infty}=0 \tag{8}$$

$$M_0u_m{}^*=-\sum_{k=1}^{m}M_{m-k+1}u^*_{k-1}, \quad u_m{}^*|_{t=0}=0, \quad u_m{}^*|_{t\to\infty}=0 \tag{9}$$

$$(m=1,\ 2,\ 3,\ldots)$$

We see that $u_0{}^*=e^{-t}$ is a solution of (8). It is also clear that the solution of (9) is a function of the form

$$u_n{}^*=\sum_{l=1}^{n}a_l^{(n)}(\varphi)\,t^l\,e^{-t} \tag{10}$$

Using (5) we rewrite (9) in the form

$$\left(\frac{\partial^2}{\partial t^2}-1\right)u_m{}^*=\sum_{k=1}^{m}\left(-\frac{1}{\gamma}\right)^{m-k+1}\left[t\frac{\partial}{\partial t}-(m-k)\frac{\partial^2}{\partial\varphi^2}\right]u^*_{k-1}$$

Setting $n=m$ and $n=k-1$ in the last equation, substituting (10) into it and comparing the terms of like power in t in the left and right-hand sides, we obtain

$$-2ma_m^{(m)}=\sum_{k=1}^{m}\frac{(-1)^{m-k}}{\gamma^{m-k+1}}a_{k-1}^{(k-1)} \tag{11}$$

$$(m-l+1)\left[(m-l)\,a_{m-l}^{(m)}-2a_{m-l-1}^{(m)}\right]=$$

$$\sum_{k=2+l}^{m}\frac{(-1)^{m-k+1}}{\gamma^{m-k+1}}\left[(k-l-1)\,a_{k-l-1}^{(k-1)}-a_{k-l-2}^{(k-1)}-(m-k)\frac{d^2a_{k-1-l}^{(k-1)}}{d\varphi^2}\right] \tag{12}$$

$$(l=0,\ 1,\ 2,\ldots)$$

From (11) we find

$$-2ma_m^{(m)}=-\frac{1}{\gamma}\sum_{k=1}^{m-1}\left(-\frac{1}{\gamma}\right)^{m-k-1}a_{k-1}^{(k-1)}+\frac{1}{\gamma}a_{m-1}^{(m-1)}=\frac{2m-1}{\gamma}a_{m-1}^{(m-1)}$$

In this manner we obtain the following recurrence relation :

$$a_m^{(m)}=\frac{1}{\gamma}\left(\frac{1}{2m}-1\right)a_{m-1}^{(m-1)}, \quad a_0^{(0)}=1 \qquad (m=1,\ 2,\ldots) \tag{13}$$

Similarly, setting in (12) $l=0,\ l=1$, etc., we obtain the recurrence relations

$$a_m^{(m+1)} = \frac{3-4m}{2\gamma m}\, a_{m-1}^{(m)} + \frac{3-2m}{2\gamma^2 m}\, a_{m-2}^{(m-1)} + \frac{1}{8\gamma^2 m}\, a_{m-1}^{(m-1)} + \frac{1}{2\gamma^2 m}\, \frac{d^2 a_{m-1}^{(m-1)}}{d\varphi^2}, \quad a_0^{(1)} = 0$$

$$a_m^{(m+2)} = \frac{3-4m}{2\gamma m}\, a_{m-1}^{(m+1)} + \frac{3-2m}{2\gamma^2 m}\, a_{m-2}^{(m)} - \frac{3}{4\gamma}\, a_m^{(m+1)} + \frac{2-3m}{4\gamma^2 m}\, a_{m-1}^{(m)} + \qquad (14)$$

$$\frac{1}{16\gamma^2 m}\, a_m^{(m)} + \frac{1}{2\gamma^2 m}\, \frac{d^2 a_{m-1}^{(m)}}{d\varphi^2} + \frac{1}{4\gamma^2}\, \frac{d^2 a_m^{(m)}}{d\varphi^2}, \quad a_0^{(2)} = 0 \quad (m = 1, 2, \ldots)$$

Taking into account (10), let us write the sum appearing in the right-hand side of (6) in the form

$$\sum_{m=0}^{\infty} \varepsilon^m u_m{}^* = \sum_{m=0}^{\infty} a_m^{(m)} \xi^m e^{-t} + \ldots + \varepsilon^l \sum_{m=l+1}^{\infty} a_{m-l}^{(m)} \xi^{m-l} e^{-t} + \ldots$$

The following expression represents the nth approximation of the exact solution u^*:

$$\left(\frac{u_0}{s}\right) \sum_{l=0}^{n} \varepsilon^l x_l e^{-t}$$

where

$$\xi = \varepsilon t, \qquad x_0 = \sum_{m=0}^{\infty} a_m^{(m)} \xi^m, \qquad x_l = \sum_{m=l+1}^{\infty} a_{m-l}^{(m)} \xi^{m-l}$$

$$(l = 1, 2, \ldots)$$

We find x_0 using (13). We have

$$\left(1 + \frac{\xi}{\gamma}\right) x_0 = 1 + \frac{1}{2\gamma} \sum_{m=1}^{\infty} \frac{1}{m}\, a_{m-1}^{(m-1)} \xi^m \qquad (15)$$

Differentiating (15) with respect to ξ, we obtain for x_0, the following ordinary differential equation

$$\frac{\partial x_0}{\partial \xi} + \frac{x_0}{2(\gamma + \xi)} = 0, \qquad x_0(0) = 1 \qquad (16)$$

from which we have

$$x_0 = (1 + \xi / \gamma)^{-1/2} = (\gamma / \rho)^{1/2}$$

Thus the zero order approximation to the solution of (3) is

$$u^* = \frac{u_0}{s} \left[\frac{a\gamma(\varphi)}{r}\right]^{1/2} e^{-[r - a\gamma(\varphi)] \sqrt{s/\varkappa}} \qquad (17)$$

Inversion of the Laplace transform gives the following zero order approximation to the solution of the problem (1), (2)

$$u = u_0 \left[\frac{a\gamma(\varphi)}{r}\right]^{1/2} \operatorname{erfc}\left(\frac{r - \alpha\gamma}{2\sqrt{\varkappa\tau}}\right) \qquad (18)$$

In the same manner we find x_1, x_2, etc. For x_1 the differential equation in ξ has the form

$$\frac{\partial x_1}{\partial \xi} + \frac{x_1}{2(\gamma + \xi)} = \frac{1}{8\gamma^2}\left(1 + \frac{\xi}{\gamma}\right)^{-2} \left(x_0 + 4\frac{\partial^2 x_0}{\partial \varphi^2}\right), \qquad x_1(0) = 0 \qquad (19)$$

from which we obtain

$$x_1 = \frac{1}{8\gamma}\left(\frac{\gamma}{\rho}\right)^{1/2} \left(1 - \frac{\gamma}{\rho}\right)\left[1 + \gamma^2\left(1 - \frac{\gamma}{\rho}\right)^2 \left(\frac{d\gamma^{-1}}{d\varphi}\right)^2 - \gamma\left(1 - \frac{\gamma}{\rho}\right)\left(\frac{d^2\gamma^{-1}}{d\varphi^2}\right)\right]$$

Thus the first order approximation to the solution of the problem (3) is

$$u^* = \left(\frac{a\gamma}{r}\right)^{1/2} \left\{1 + \frac{1}{8a\gamma}\left(1 - \frac{a\gamma}{r}\right)\left[1 + \gamma^2\left(1 - \frac{a\gamma}{r}\right)^2 \left(\frac{d\gamma^{-1}}{d\varphi}\right)^2 - \right.\right.$$

$$\left.\left. \gamma\left(1 - \frac{a\gamma}{r}\right)\frac{d^2\gamma^{-1}}{d\varphi^2}\right] \sqrt{\frac{\varkappa}{s}}\right\} \frac{u_0}{s}\, e^{-[r - a\gamma(\varphi)] \sqrt{s/\varkappa}} \qquad (20)$$

Inverting the Laplace transform gives the first order approximation to the solution of (1), (2)

$$u = u_0\left(\frac{a\gamma}{r}\right)^{1/2}\left\{\operatorname{erfc}\left(\frac{r-a\gamma}{2\sqrt{\varkappa\tau}}\right)+\frac{\sqrt{\varkappa\tau}}{4a\gamma}\left(1-\frac{a\gamma}{r}\right)\left[1+\gamma^2\left(1-\frac{a\gamma}{r}\right)^2\left(\frac{d\gamma^{-1}}{d\varphi}\right)^2-\right.\right.$$
$$\left.\left.\gamma\left(1-\frac{a\gamma}{r}\right)\frac{d^2\gamma^{-1}}{d\varphi^2}\right]\operatorname{ierfc}\left(\frac{r-a\gamma}{2\sqrt{\varkappa\tau}}\right)\right\} \qquad (21)$$

The first order approximation formulas can be rewritten more simply. Let the temperature be defined at the point M and let a point P lie on Γ such, that the segment MP is orthogonal to Γ. If $R = R_P$ is the radius of curvature of Γ at the point P and d_{MP} is the distance between M and P, then the expressions (20) and (21) become

$$u^* = \left(1+\frac{d_{MP}}{R_P}\right)^{-1/2}\left[1+\frac{d_{MP}}{8R_P\,(R_P+d_{MP})}\sqrt{\frac{\varkappa}{s}}\right]\frac{u_0}{s}e^{-d_{MP}\sqrt{s/\varkappa}}$$

$$u = u_0\left(1+\frac{d_{MP}}{R_P}\right)^{-1/2}\left[\operatorname{erfc}\left(\frac{d_{MP}}{2\sqrt{\varkappa\tau}}\right)+\frac{\sqrt{\varkappa\tau}\,d_{MP}}{4R_P\,(R_P+d_{MP})}\operatorname{ierfc}\left(\frac{d_{MP}}{2\sqrt{\varkappa\tau}}\right)\right] \qquad (22)$$

With the equations for x_0 and x_1 taken into account, the differential equation in ξ for x_2 has the form

$$\frac{\partial x_2}{\partial\xi}+\frac{x_2}{2(\gamma+\xi)}=\frac{1}{8\gamma^2}\left(1+\frac{\xi}{\gamma}\right)^{-2}\left[x_1+4\frac{\partial^2 x_1}{\partial\varphi^2}-\frac{x_0}{\gamma+\xi}-\frac{3}{\gamma+\xi}\frac{\partial^2 x_0}{\partial\varphi^2}-\frac{\partial^2}{\partial\varphi^2}\left(\frac{x_0}{\gamma+\xi}\right)\right]$$
$$x_2(0)=0 \qquad (23)$$

Choosing the arc length λ as the parameter and attaching to the point P the value $\lambda = \lambda_P$, we can write the solution of (23) in the form (24)

$$x_2 = a^2\left(1+\frac{d_{MP}}{R_P}\right)^{-1/2}\left[-\frac{d_{MP}\,(7d_{MP}+16R_P)}{128R_P^2\,(R_P+d_{MP})^2}-\frac{d_{MP}^3\,R_P}{48\,(R_P+d_{MP})^3}\left(\frac{d^2R^{-1}}{d\lambda^2}\right)_{\lambda=\lambda_P}\right]$$

Thus the second order approximation to the solution of (3) is

$$u^* = u_0\left(1+\frac{d_{MP}}{R_P}\right)^{-1/2}\left\{1+\frac{d_{MP}}{8R_P\,(R_P+d_{MP})}\sqrt{\frac{\varkappa}{s}}-\left[\frac{d_{MP}\,(7d_{MP}+16R_P)}{128R_P^2\,(R_P+d_{MP})^2}+\right.\right.$$
$$\left.\left.\frac{d_{MP}^3\,R_P}{48\,(R_P+d_{MP})^3}\left(\frac{d^2R^{-1}}{d\lambda^2}\right)_{\lambda=\lambda_P}\right]\frac{\varkappa}{s}\right\}\frac{1}{s}\,e^{-d_{MP}\sqrt{s/\varkappa}} \qquad (25)$$

Inversion of the Laplace transform yields the second order approximation to the solution of (1), (2)

$$u = u_0\left(1+\frac{d_{MP}}{R_P}\right)^{-1/2}\left\{\operatorname{erfc}\left(\frac{d_{MP}}{2\sqrt{\varkappa\tau}}\right)+\frac{\sqrt{\varkappa\tau}\,d_{MP}}{4R_P\,(R_P+d_{MP})}\operatorname{ierfc}\left(\frac{d_{MP}}{2\sqrt{\varkappa\tau}}\right)-\right. \qquad (26)$$
$$\left.\frac{\varkappa\tau}{4R_P^2}\left[\frac{d_{MP}\,(16+7d_{MP}/R_P)}{8R_P\,(1+d_{MP}/R_P)^2}+\frac{d_{MP}^3}{3\,(1+d_{MP}/R_P)^3}\left(\frac{d^2R^{-1}}{d\lambda^2}\right)_{\lambda=\lambda_P}\right]i^2\operatorname{erfc}\left(\frac{d_{MP}}{2\sqrt{\varkappa\tau}}\right)\right\}$$

Setting now in (25) and (26) $R_P = a = \text{const}$ we find, that $d_{MP} = r - a$ and thus arrive at the known second order approximation formulas for the problem of propagation of heat in a region bounded within by a circle [3].

As an example, we consider the case when the contour Γ is an ellipse with the semi-axes equal to a and $\frac{1}{2}a$, defined by the equations $x = a\cos t$ and $y = \frac{1}{2}a\sin t$. Let the value $t = t_P$ correspond to the point P. Figure 1 shows the temperature distributions along the ray PM orthogonal to Γ at the point P, for the values of time corresponding to $\varkappa\tau / a^2 = 0.04$ (curves *1*) and $\varkappa\tau / a^2 = 0.09$ (curves *2*), and the values of t_p equal to 0(a), $\pi / 4$(b) and $\pi / 2$ (c).

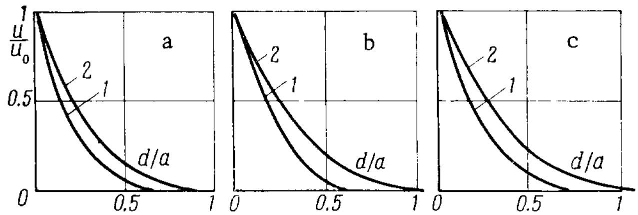

Fig. 1

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DETERMINATION OF THE FREQUENCY OF THE APPROXIMATE SOLUTION OF HILL'S EQUATION

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In the theory of motion of charged particles through periodic focussing accelerators the Hill's equation is often solved using a widely accepted method of "smooth approximation". By this method the solution is represented in the form of a "slow" harmonic function with a "rapidly" oscillating amplitude. Below we derive a formula for the frequency of the slow component of such a solution, expressed in terms of the Fourier harmonics of the equation coefficient. Such a formula may find use in practical computations.

In the smooth approximation [1] which converges to the first approximation of the method of averaging [2] the solution of the Hill equation

$$x^{\cdot\cdot} + q(t)x = 0, \quad q(t + T) \equiv q(t) \qquad (T > 0) \tag{1}$$

is sought in the form $x(t) = [1 + r(t)]X(t)$, where $X(t)$ represents a slow (compared